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CITATION:

黒木, 慎太郎. A remark on torus graph with root systems of type A (New topics of transformation groups). 数理解析研究所講究録 2015, 1968: 55-58: KJ00010055447.

ISSUE DATE:

2015-11

URL:

<http://hdl.handle.net/2433/224277>

RIGHT:

# A remark on torus graph with root systems of type A

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## 1. Introduction

In the previous paper [KuMa], we define a root system on a torus manifold, and characterize extended actions of torus manifolds. Due to the work of Maeda-Masuda-Panov [MMP], there is a combinatorial counterpart of torus manifold, called a *torus graph*  $(\Gamma, \mathcal{A})$ . Here,  $\Gamma = (V(\Gamma), E(\Gamma))$  is an abstract  $n$ -valent graph and  $\mathcal{A} : E(\Gamma) \rightarrow H^2(BT^n)$  is a label on edges, called an *axial function*. Therefore, we can also define *root systems* on torus graphs like torus manifolds. In this article, we characterize the torus graph with root systems of type A combinatorially.

## 2. Root systems of type A of torus graphs

Let  $(\Gamma, \mathcal{A})$  be a torus graph, and  $H_T^*(\Gamma, \mathcal{A})$  be its *graph equivariant cohomology*, i.e.,  

$$H_T^*(\Gamma, \mathcal{A}) := \{f : V(\Gamma) \rightarrow H^*(BT^n) \mid f(p) \equiv f(q) \pmod{\mathcal{A}(e)} \text{ for } i(e) = p, t(e) = q\},$$
where  $i(e)$  (resp.  $t(e)$ ) is the initial (resp. terminal) vertex of  $e \in E(\Gamma)$ . Then, we can define the following injective homomorphism

$$\varphi : H^*(BT^n) \rightarrow H_T^*(\Gamma, \mathcal{A})$$

by

$$\varphi(\alpha) := \alpha,$$

where  $\alpha : V(\Gamma) \rightarrow H^*(BT^n)$  is the constant map, i.e.,  $\alpha(p) = \alpha$  for all  $p \in V(\Gamma)$ . Then, with the method similar to define a root system of type A of torus manifold in [KuMa], we can define a *root system of type A* on torus graph as follows.

**DEFINITION 2.1.** We call the set  $R(\Gamma, \mathcal{A}) \subset H^2(BT^n)$  a *root system of type A* of a torus graph  $(\Gamma, \mathcal{A})$  if  $\alpha \in R(\Gamma, \mathcal{A})$  then  $-\alpha \in R(\Gamma, \mathcal{A})$  and  $\varphi(\alpha) = \tau_i - \tau_j$  for some Thom classes  $\tau_i$  and  $\tau_j$  of  $(n-1)$ -valent torus subgraphs  $\Gamma_i$  and  $\Gamma_j$ .

**PROPOSITION 2.2.** The above  $R(\Gamma, \mathcal{A})$  satisfies the axiom of root systems in [Hu] with respect to the inner product of  $H^2(BT^n)$  defined by the pairing with  $H_2(BT^n)$  (see [KuMa]).

Let  $\Delta(\Gamma, \mathcal{A}) = \{\alpha_1, \dots, \alpha_\ell\}$  be a simple root of  $R(\Gamma, \mathcal{A})$ . If there exists a string  $\tau_1, \dots, \tau_{\ell+1}$  of Thom classes such that  $\varphi(\alpha_i) = \tau_i - \tau_{i+1}$  for all  $i = 1, \dots, \ell$ , then  $R(\Gamma, \mathcal{A})$  is called an *irreducible*.

### 3. Main theorem

In order to state the main theorem, we need to prepare some notations.

**3.1. Fibration of torus graphs.** We first recall the fibration of torus graphs (also see [GSZ]).

Let  $\Gamma$  and  $B$  be connected graphs and  $\rho : \Gamma \rightarrow B$  be a morphism of graphs. Hence  $\rho$  is a map from the vertices of  $\Gamma$  to the vertices of  $B$  such that if  $pq \in E(\Gamma)$  then either  $\rho(p) = \rho(q)$  or else  $\rho(p)\rho(q) \in E(B)$ . If  $pq \in E(\Gamma)$  and  $\rho(p) = \rho(q)$  then we will say that the edge  $pq$  is *vertical*, and if  $\rho(p)\rho(q) \in E(B)$  then we will say that the edge  $pq$  is *horizontal*. For a vertex  $q \in V(\Gamma)$ , let  $E_q^\perp(\Gamma)$  be the set of vertical edges with initial vertex  $q$ , and let  $H_q(\Gamma)$  be the set of horizontal edges with initial vertex  $q$ . Then  $E_q(\Gamma) = E_q^\perp(\Gamma) \cup H_q(\Gamma)$  and  $\rho$  induces canonically a map

$$(d\rho)_q : H_q(\Gamma) \rightarrow E_{\rho(q)}(B)$$

from the horizontal edges at  $q$  to the edges of  $B$  with initial vertex  $\rho(q)$ : if  $qq' \in H_q(\Gamma)$ , then  $(d\rho)_q(qq') = \rho(q)\rho(q')$ .

**DEFINITION 3.1.** The morphism of graphs  $\rho : \Gamma \rightarrow B$  is a *fibration* of graphs if for every vertex  $q$  of  $\Gamma$ , the map  $(d\rho)_q : H_q(\Gamma) \rightarrow E_{\rho(q)}(B)$  is bijective.

Let us define the fibration of torus graphs.

**DEFINITION 3.2.** Let  $(\Gamma, \mathcal{A})$  and  $(B, \mathcal{A}_B)$  be torus graphs. A morphism  $\rho : (\Gamma, \mathcal{A}) \rightarrow (B, \mathcal{A}_B)$  is a *fibration* of torus graphs, if it satisfies the following conditions:

- (1)  $\rho : \Gamma \rightarrow B$  is a fibration of graphs;
- (2) If  $e$  is an edge of  $B$  and  $\tilde{e}$  is any lift of  $e$ , then  $\mathcal{A}(\tilde{e}) = \mathcal{A}_B(e)$ .

Comparing with the definition of GKM-fibrations in [GSZ] (also see [Ku]), we do not need to assume the compatible conditions of connections. This is because the connections of torus graphs are uniquely determined. In particular, we have the following proposition.

**PROPOSITION 3.3.** Let  $\rho : (\Gamma, \mathcal{A}) \rightarrow (B, \mathcal{A}_B)$  be a fibration of torus graphs. Assume that  $\Gamma$  is  $n$ -valent and  $B$  is  $\ell$ -valent. Then, for all  $p \in V(B)$ ,  $\rho^{-1}(p)$  is an  $(n - \ell)$ -valent torus subgraph of  $\Gamma$ .

**3.2. Blow-up of torus graphs.** We next introduce a blow-up of a torus graph (see [MMP]).

Let  $(\Gamma', \mathcal{A}')$  be an  $(n - \ell)$ -valent torus subgraph of the  $n$ -valent GKM graph  $(\Gamma, \mathcal{A})$ . Then, the cardinality of the normal edges  $N_p(\Gamma')$  is exactly  $\ell$ ; therefore, we may denote  $N_p(\Gamma') = \{pp'_1, \dots, pp'_\ell\}$ .

The *blow-up* of  $\Gamma$  along  $\Gamma'$ , denoted  $\tilde{\Gamma} = (V(\tilde{\Gamma}), E(\tilde{\Gamma}))$ , is defined as follows. The vertex set is defined as  $V(\tilde{\Gamma}) = (V(\Gamma) - V(\Gamma')) \cup V(\Gamma')^\ell$ , where  $V(\Gamma')^\ell = V(\Gamma') \times \dots \times V(\Gamma')$ .

( $\ell$  times Cartesian product), i.e., the vertex  $p \in V(\Gamma') \subset V(\Gamma)$  is replaced by  $\ell$  vertices  $\tilde{p}_1, \dots, \tilde{p}_\ell$ . It is convenient to regard those points as chosen close to  $p$  on edges from  $N_p(\Gamma') = \{pp'_1, \dots, pp'_\ell\}$ , i.e.,  $\tilde{p}_i \in pp'_i$ . Then the edges and the corresponding values of the axial function  $\tilde{\mathcal{A}}: E(\tilde{\Gamma}) \rightarrow H^2(BT)$  are defined as follows:

- (1)  $\tilde{p}_i\tilde{p}_j \in E(\tilde{\Gamma})$  for every  $p \in V(\Gamma')$ ;  $\tilde{\mathcal{A}}(\tilde{p}_i\tilde{p}_j) = \mathcal{A}(pp'_j) - \mathcal{A}(pp'_i)$ ;
- (2)  $\tilde{p}_i\tilde{q}_i \in E(\tilde{\Gamma})$  if  $pq \in E(\Gamma')$ ;  $\tilde{\mathcal{A}}(\tilde{p}_i\tilde{q}_i) = \mathcal{A}(pq)$ ;
- (3)  $\tilde{p}_ip'_i \in E(\tilde{\Gamma})$  for every  $p \in V(\Gamma')$ ;  $\tilde{\mathcal{A}}(\tilde{p}_ip'_i) = \mathcal{A}(pp'_i)$ ;
- (4) edges “coming from  $\Gamma$ ”, that is,  $pq \in E(\Gamma)$  such that  $p, q \notin V(\Gamma')$ ;  $\tilde{\mathcal{A}}(pq) = \mathcal{A}(pq)$ .

Combinatorially, this operation is nothing but the gluing of  $\Gamma' \times K_{\ell+1}$  along the subgraph  $\Gamma' \subset \Gamma$ , where  $K_{\ell+1}$  is the complete graph with  $(\ell+1)$ -vertices, i.e.,  $V(K_{\ell+1}) = \{p_0, \dots, p_\ell\}$ ,  $E(K_{\ell+1}) = \{p_ip_j \mid i \neq j\}$ .

The following proposition is straightforward.

**PROPOSITION 3.4.** Let  $(\Gamma, \mathcal{A})$  be an  $n$ -valent torus graph and  $(\Gamma', \mathcal{A}')$  be a torus subgraph. Then, its blow-up  $(\tilde{\Gamma}, \tilde{\mathcal{A}})$  along  $(\Gamma', \mathcal{A}')$  is an  $n$ -valent torus graph. Moreover, there is the natural morphism from  $(\tilde{\Gamma}, \tilde{\mathcal{A}})$  to  $(\Gamma, \mathcal{A})$ .

**3.3. Main theorem.** The main theorem can be stated as follows:

**THEOREM 3.5.** Let  $(\Gamma, \mathcal{A})$  be a torus graph. Suppose that there exists an irreducible non-empty root system of type A, say  $R(\Gamma, \mathcal{A})$ . Choose its simple root as  $\Delta(\Gamma, \mathcal{A}) = \{\alpha_1, \dots, \alpha_\ell\} \in H^2(BT^n)$  such that  $\varphi(\alpha_i) = \tau_i - \tau_{i+1}$  for  $i = 1, \dots, \ell$ , where  $\tau_i$  is the Thom class of the  $(n-1)$ -valent torus subgraph  $\Gamma_i$ . Then, one of the following cases occur:

**The 1<sup>st</sup> case:** if  $\tau_1 \cdots \tau_{\ell+1} = 0$  and  $\cap_{i \in I} \tau_i \neq 0$  for all  $I \subset [\ell+1]$  with  $|I| = \ell$ , i.e.,  $\Gamma_1 \cap \cdots \cap \Gamma_{\ell+1} = \emptyset$  but  $\cap_{i \in I} \Gamma_i \neq \emptyset$ , then there is the fibration

$$\rho: (\Gamma, \mathcal{A}) \rightarrow (K_{\ell+1}, \mathcal{A}_{\ell+1});$$

**The 2<sup>nd</sup> case:** otherwise, i.e.,  $\Gamma_1 \cap \cdots \cap \Gamma_{\ell+1} \neq \emptyset$ , there is the blow-up  $(\tilde{\Gamma}, \tilde{\mathcal{A}}) \rightarrow (\Gamma, \mathcal{A})$  along  $\Gamma' = \Gamma_1 \cap \cdots \cap \Gamma_{\ell+1}$  such that  $(\tilde{\Gamma}, \tilde{\mathcal{A}})$  satisfies the 1<sup>st</sup> case.

In the statement of theorem,  $\mathcal{A}_{\ell+1}$  is the standard axial function of the complete graph  $K_{\ell+1}$  which defined by  $\mathcal{A}_{\ell+1}(p_0p_j) = \alpha_j$  and  $\mathcal{A}_{\ell+1}(p_ip_j) = \alpha_j - \alpha_i$  for  $i, j \neq 0$ . Namely,  $(K_{\ell+1}, \mathcal{A}_{\ell+1})$  is the torus graph which is obtained by the standard  $T^n$ -action on  $\mathbb{CP}^n$ .

**REMARK 3.6.** Note that in [Ku] we announced an analogues result for all GKM graphs with root systems of type A. However, in general, the GKM blow-up is not well-defined for GKM graphs. So we need to change the statement of the main theorem in [Ku] as above.

### Acknowledgment

This work was supported by JSPS KAKENHI Grant Number 15K17531, 24224002.

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